11. Find $f(25)$ given $f(20) = 14$, $f(24) = 32$, $f(28) = 35$ and $f(32) = 40$, using Gauss's formula.

## 7.5. Stirling's formula

By Gauss's forward formula, we have

$$y(x) = y(x_0 + uh) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \cdots \quad ...(1)$$

By Gauss's backward formula, we have

$$y(u) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \cdots \quad ...(2)$$

Adding (1) and (2),

$$2y(u) = 2y_0 + \binom{u}{1}[\Delta y_0 + \Delta y_{-1}] + \left[\binom{u}{2} + \binom{u+1}{2}\right]\Delta^2 y_{-1} + \binom{u+1}{3}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \left[\binom{u+2}{4} + \binom{u+1}{4}\right]\Delta^4 y_{-2} + \cdots$$

$$= 2y_0 + \binom{u}{1}[\Delta y_0 + \Delta y_{-1}] + \frac{2u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{2u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$\therefore \quad \mathbf{y(x) = y(x_0 + uh) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right)}{2} + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots} \quad ...(3)$$

where $u = \dfrac{x - x_0}{h}$.

Equation (3) is known as *Stirling's formula* and it is the average of the two Gauss's formulae.

**Note 1.** The formula involves the means of the *odd* differences just above and just below the central line and even differences on the central line.

2. The formula can be remembered with the help of the following table.

| | | | | | |
|---|---|---|---|---|---|
| Coefficient : | $1$ | $u$ | $\frac{u^2}{2}$ | $\frac{u(u^2-1^2)}{3!}$ | $\frac{u^2(u^2-1^2)}{4!}$ |
| Differences : | $y_0$ | $\frac{1}{2}(\Delta y_0 + \Delta y_{-1})$ | $\Delta^2 y_{-1}$ | $\frac{1}{2}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right)$ | $\Delta^4 y_{-2}$ |

That is, (diagrammatically) the differences in the terms are

$$y_0 \;\text{———}\; \begin{pmatrix} \Delta y_{-1} \\ \\ \Delta y_0 \end{pmatrix} \;\text{—}\; \Delta^2 y_{-1} \;\text{—}\; \begin{pmatrix} \Delta^3 y_{-2} \\ \\ \Delta^2 y_{-1} \end{pmatrix} \;\text{—}\; \Delta^4 y_{-2}$$

$y_1$

Average                    Average

3. To use this formula, we must have $-\frac{1}{2} < u < \frac{1}{2}$.

## 7.6. Bessel's formula

By Gauss's forward formula, we get

$$y(x) = y(x_0 + uh) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \cdots \quad ...(1)$$

We know, $\Delta y_0 = y_1 - y_0$

$$\therefore \quad y_0 = y_1 - \Delta y_0 \quad ...(i)$$

$$y_{-1} = y_0 - \Delta y_{-1}$$

$$\therefore \quad \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \quad ...(ii)$$

Also, $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc.

Hence, (1) is rewritten as,

$$y(x) = y(x_0 + uh) = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + u\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!} \cdot \Delta^2 y_{-1} + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)(u)(u-1)}{3!}\Delta^3 y_{-1} + \cdots \quad ...(2)$$

using (*i*) and (*ii*) in (2)

$$y(x_0 + uh) = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + u\,\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\cdot\frac{u(u-1)}{2!}\left(\Delta^2 y_0 - \Delta^3 y_{-1}\right) + \frac{(u+1)\,u\,(u-1)}{3!}\Delta^3 y_{-1} +$$

$$= \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \frac{u(u-1)}{2!}\left(-\frac{1}{2} + \frac{u+1}{3}\right)\Delta^3 y_{-1} +$$

$$y(x) = y(x_0 + uh) = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots \quad \ldots(3)$$

Equation (3) is known as Bessel's formula.

**Note 1.** The formula involves *odd* differences below the central line and means of the even differences on and below the central line.

| | | | | | |
|---|---|---|---|---|---|
| Central line | $y_0$ | $\Delta y_0$ | $\Delta^2 y_{-1}$ | $\Delta^3 y_{-1}$ | $\Delta^4 y_{-2}$ |
| Next line | $y_1$ | | $\Delta^2 y_0$ | | $\Delta^4 y_{-1}$ |
| | average | | average | | average |

If $u = \frac{1}{2}$, the coefficients of all odd order differences are zero.

Hence, setting $u = \frac{1}{2}$, we have

$$y\left(u = \frac{1}{2}\right) = \frac{y_0 + y_1}{2} - \frac{1}{8}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3}{128}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$- \frac{5}{1024}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots$$

**Note 1.** This form is suited to compute the values of the function midway between two given values.

**2.** Only even order differences exist in the formula.

This is also known as *formula for interpolating to halves.*

## 7·7. Laplace-Everett formula

Gauss's forward formula is

$$y(x) = y(x_0 + uh) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1}$$

$$+ \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \cdots \quad \ldots(1)$$

We have $\Delta y_0 = y_1 - y_0$

$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$

$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ etc.

Substituting these in (1), we have

$$y(x_0 + uh) = y_0 + \binom{u}{1}(y_1 - y_0) + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\left(\Delta^2 y_0 - \Delta^2 y_{-1}\right)$$

$$+ \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\left(\Delta^4 y_{-1} - \Delta^4 y_{-2}\right) + \cdots$$

3. If interpolation is required near the middle values of the table, use either Stirling's or Bessel's formula.

4. If $-\frac{1}{4} < u \le \frac{1}{4}$, then use Stirling's formula.

5. If $\frac{1}{4} < u < \frac{3}{4}$, then use Bessel's formula for better results.

**Example 1.** *Using Stirlings formula, find y (1·22) from the following table.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *x* | *1·0* | *1·1* | *1·2* | *1·3* | *1·4* | *1·5* | *1·6* |
| *y* | *0·84147* | *0·89121* | *0·93204* | *0·96356* | *0·98545* | *0·99749* | *0·99957* |
| | *1·7* | *1·8* | | | | | |
| | *0·99385* | *0·97385* | | | | | |

*(Delhi Uni.)*

**Solution.** Since we require $y$ at $x = 1{\cdot}22$, take the origin at $x = 1{\cdot}2$ and $h = 0{\cdot}1$

$$u = \frac{x - x_0}{h} = \frac{1{\cdot}22 - 1{\cdot}2}{0{\cdot}1} = \frac{0{\cdot}02}{0{\cdot}1} = 0{\cdot}2$$

We form the central difference table below. Since $x = 1{\cdot}2$ is the origin, we take values on both sides of $1{\cdot}2$ to the required stage.

**Difference Table**

| $x$ | $u$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| 1·0 | −2 | 0·84147 | | | | |
| | | | 0·04974 | | | |
| | | | | −0·00891 | | |
| 1·1 | −1 | 0·89121 | | | −0·00040 $\Delta^3 y_{-2}$ | |
| | | | 0·04083 $\Delta y_{-1}$ | | | |
| 1·2 | 0 | 0·93204 $y_0$ | | −0·00931 $\Delta^2 y_{-1}$ | | 0·00008 $\Delta^4 y_{-2}$ |
| | | | 0·03152 $\Delta y_0$ | | | |
| 1·3 | 1 | 0·96356 | | | −0·00032 $\Delta^3 y_{-1}$ | |
| | | | | −0·00963 | | |
| | | | 0·02189 | | | |
| 1·4 | 2 | 0·98545 | | | | |

By Stirling's formula, we have

$$y(x_0 + uh) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1}$$

$$+ \frac{u(u^2 - 1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2 - 1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(1{\cdot}2) = 0{\cdot}93204 + (0{\cdot}2)\left[\frac{0{\cdot}04083 + 0{\cdot}03152}{2}\right] + \frac{(0{\cdot}2)^2}{2}(-0{\cdot}00931)$$

$$+ \frac{(0{\cdot}2)(0{\cdot}04 - 1)}{6}\left[\frac{-0{\cdot}00040 - 0{\cdot}00032}{2}\right] + \frac{(0{\cdot}04)(0{\cdot}04 - 1)}{24}(0{\cdot}00008) + \cdots$$

$$= 0{\cdot}93204 + 0{\cdot}007235 - 0{\cdot}0001862 + {\cdot}00001152 - 0{\cdot}000000128$$

$$= \mathbf{0{\cdot}939100192}$$

**Example 2.** *From the following table, estimate $e^{0.644}$ correct to five decimals using (i) Stirling's formula (ii) Bessel's formula (iii) Everett's formula.*

*Also find $e^x$ at $x = 0.638$.*

| $x$ | 0·61 | 0·62 | 0·63 | 0·64 | 0·65 | 0·66 | 0·67 |
|---|---|---|---|---|---|---|---|
| $e^x$ | 1·840431 | 1·858928 | 1·877610 | 1·896481 | 1·915541 | 1·934792 | 1·954237 |

(Anna Nov. 2004)

Take $x = 0.64$ as the origin,

$$u = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4.$$

**Difference Table**

| $x$ | $u$ | $y = e^x$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| 0·61 | – 3 | 1·840431 | | | | |
| | | | 0·018497 | | | |
| 0·62 | – 2 | 1·858928 | | 0·000185 | | |
| | | | 0·018682 | | 0·000004 | |
| 0·63 | – 1 | 1·877610 | | 0·000189 | | – 0·000004 |
| | | | 0·018871 | | 0·0 | |
| 0·64 | 0 | 1·896481 | | 0·000189 | | 0·000002 |
| | | | 0·019060 | | 0·000002 | |
| 0·65 | 1 | 1·915541 | | 0·000191 | | 0·000001 |
| | | | 0·019251 | | 0·000003 | |
| 0·66 | 2 | 1·934792 | | 0·000194 | | |
| | | | 0·019445 | | | |
| 0·67 | 3 | 1·954237 | | | | |

(*i*) By Stirling's formula,

$$y(x = 0.644) = y(u = 0.4) = 1.896481 + (0.4)\left(\frac{0.018871 + 0.019060}{2}\right)$$

$$+ \frac{0.16}{2}(0.000189) + \frac{(0.4)(0.16 - 1)}{6}\left(\frac{0 + 0.000002}{2}\right) + \cdots$$

$$= 1.896481 + 0.0075862 + 0.00001512 - 0.000000056$$

$$= \mathbf{1.90408226}$$

(*ii*) Using Bessel's formula, we get

$$y(0.644) = \frac{1.896481 + 1.915541}{2} + \left(0.4 - \frac{1}{2}\right)(0.019060)$$

$$+ \frac{(0.4)(-0.6)}{2}\left(\frac{0.000189 + 0.000191}{2}\right)$$

$$+ \frac{(0.4)(-0.6)(0.4 - 0.5)}{6}(0.000002) + \cdots$$

$$= 1.906011 - 0.001906 - 0.0000228$$

$$= 1.904082$$